ME-221
SOLUTIONS FOR PROBLEM SET 7

Problem 1

a)

$$X(s) = \frac{s(s+1)}{(s+2)(s+3)(s+4)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

With:

$$A = \lim_{s\to -2}\{(s+2)X(s)\} = 1$$
$$B = \lim_{s\to -3}\{(s+3)^2X(s)\} = -6$$
$$C = \lim_{s\to -4}\{(s+4)^2X(s)\} = 6$$

Hence,

$$x(t) = e^{-2t} - 6e^{-3t} + 6e^{-4t} \qquad for \quad t \geq 0$$

b)

$$X(s) = \frac{s+4}{(s+1)^2} = \frac{(s+1)+3}{(s+1)^2} = \frac{1}{s+1} + \frac{3}{(s+1)^2}$$

Hence,

$$x(t) = e^{-t} + 3te^{-t} \qquad for \quad t \geq 0$$

Or,

$$X(s) = \frac{s+4}{(s+1)^2} = \frac{A_1}{(s+1)^2} + \frac{A_2}{s+1}$$

With,

$$A_1 = \lim_{s\to -1}(\frac{s+4}{(s+1)^2}(s+1)^2) = 3$$
$$A_2 = \lim_{s\to -1}\frac{d}{ds}(\frac{s+4}{(s+1)^2}(s+1)^2) = 1$$
$$x(t) = e^{-t} + 3te^{-t} \qquad for \quad t \geq 0$$

c)

$$
\begin{aligned}
X(s) &= \frac{2}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)} \\
A &= \lim_{s\to -1} \frac{d}{ds}(\frac{2}{s+2}) = \lim_{s\to -1} \frac{-2}{(s+2)^2} = -2 \\
B &= \lim_{s\to -1} \frac{2}{s+2} = 2 \\
C &= \lim_{s\to -2} \frac{2}{(s+1)^2} = 2 \\
x(t) &= -2e^{-t} + 2te^{-t} + 2e^{-2t} \ \ t \geq 0
\end{aligned}
$$

d)

$$
X(s) = \frac{5e^{-2s}}{s^3+2s^2+5s} = \frac{5}{s^3+2s^2+5s}e^{-2s} = F(s)e^{-2s}
$$

with :

$$
\begin{aligned}
F(s) &= \frac{1}{s}\frac{5}{s^2+2s+5} = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+2^2} = \frac{1}{s} - \frac{s+2}{(s+1)^2+2^2} \\
&= \frac{1}{s} - \frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}\frac{2}{(s+1)^2+2^2} \\
\rightarrow f(t) &= 1 - e^{-t}\big(cos2t + 0.5sin2t\big) \ \ t \geq 0
\end{aligned}
$$

From Laplace transform tables, we know that the e^{-2s} term in the Laplace domain corresponds to a temporal shift of $t = 2$ in the time domain, therefore we can write:

$$
\begin{aligned}
X(s) &= F(s)e^{-2s} \\
\rightarrow x(t) &= f(t-2) = 1 - e^{-(t-2)}\big(cos(2(t-2)) + 0.5sin(2(t-2))\big) \ \ t \geq 2
\end{aligned}
$$

Problem 2

The dynamical system is given by:

$$
\ddot{y}(t) + 7\dot{y}(t) + 6y(t) = u(t)
$$

The initial conditions are: $y(0) = 1$ and $\dot{y}(0) = 2$.

Remark : Here, we will calculate the total response as the sum of the forced and free responses. We could also calculate the total response directly from the Laplace Transform of the full dynamic equation, considering both the non-zero initial conditions and the input $u(t)$.

$\rightarrow$ *Calculate the forced response $y_1(t)$ of the system for a step input and zero initial conditions*

$$\begin{aligned} s^2Y_1(s) + 7sY_1(s) + 6Y_1(s) &= U(s) \\ \frac{Y_1(s)}{U(s)} &= \frac{1}{s^2+7s+6} \end{aligned}$$

For $u(t) = \epsilon(t), U(s) = 1/s$ and hence:

$$Y_1(s) = \frac{1}{s^2+7s+6}\frac{1}{s} = \frac{1}{s(s+1)(s+6)}$$

Partial fraction expansion results in the following format:

$$Y_1(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+6}$$

Residues A, B, and C can be calculated as

$$\begin{aligned} A &= \lim_{s\to 0}\{sY_1(s)\} = 1/6 \\ B &= \lim_{s\to -1}\{(s+1)Y_1(s)\} = -1/5 \\ C &= \lim_{s\to -6}\{(s+6)Y_1(s)\} = 1/30 \end{aligned}$$

Finally, we obtain the *forced response* of the system:

$$y_1(t) = \mathcal{L}^{-1}\big[Y_1(s)\big] = \frac{1}{6} - \frac{1}{5}e^{-t} + \frac{1}{30}e^{-6t} \qquad for \quad t \geq 0$$

$\rightarrow$ *Calculate the free response $y_2(t)$ of the system due to non-zero initial conditions*

$$\begin{aligned} \mathcal{L}\big[\dot{y}_2(t)\big] &= sY_2(s) - y_2(0) = sY_2(s) - 1 \\ \mathcal{L}\big[\ddot{y}_2(t)\big] &= s^2Y_2(s) - sy_2(0) - \dot{y}_2(0) = s^2Y_2(s) - s - 2 \end{aligned}$$

Partial fraction expansion results in the following format:

$$\begin{aligned} [s^2Y_2(s) - s - 2] + 7[sY_2(s) - 1] + 6Y_2(s) &= 0 \\ Y_2(s) = \frac{s+9}{(s+1)(s+6)} &= \frac{A}{s+1} + \frac{B}{s+6} \end{aligned}$$

Residues A and B can be calculated as

$$\begin{aligned} A &= \lim_{s\to -1}\{(s+1)Y_2(s)\} = 8/5 \\ B &= \lim_{s\to -6}\{(s+6)Y_2(s)\} = -3/5 \end{aligned}$$

Finally, we obtain the *free response* of the system:

$$y_2(t) = \mathcal{L}^{-1}\big[Y_2(s)\big] = \frac{8}{5}e^{-t} - \frac{3}{5}e^{-6t} \qquad for \quad t \geq 0$$

Therefore, the *total response* of the system is:

$$y(t) = y_1(t) + y_2(t) = \frac{1}{6} + \frac{7}{5}e^{-t} - \frac{17}{30}e^{-6t} \qquad for \quad t \geq 0$$

b) We can write the input as:

$$u(t) = \varepsilon(t) - \varepsilon(t-5)$$
$$\rightarrow U(s) = \frac{1}{s}(1 - e^{-5s})$$

We can notice here that using the solution of $y(t)$ (total response) found in a) and the time-shifting property, we can quickly write the output $y'(t)$ as:

$$\begin{aligned} y'(t) &= y_1(t) - y_1(t-5) + y_2(t) \\ &= \varepsilon(t)(\frac{1}{6} - \frac{1}{5}e^{-t} + \frac{1}{30}e^{-6t}) - \varepsilon(t-5)(\frac{1}{6} - \frac{1}{5}e^{-(t-5)} + \frac{1}{30}e^{-6(t-5)}) + \varepsilon(t)(\frac{8}{5}e^{-t} - \frac{3}{5}e^{-6t}) \end{aligned}$$

Which can also be written as:

For $0 \leq t < 5$:

$$\begin{aligned} y'(t) &= 1(\frac{1}{6} - \frac{1}{5}e^{-t} + \frac{1}{30}e^{-6t}) - 0(\frac{1}{6} - \frac{1}{5}e^{-(t-5)} + \frac{1}{30}e^{-6(t-5)}) + 1(\frac{8}{5}e^{-t} - \frac{3}{5}e^{-6t}) \\ &= \frac{1}{6} + \frac{7}{5}e^{-t} - \frac{17}{30}e^{-6t} \end{aligned}$$

For $t \geq 5$:

$$\begin{aligned} y'(t) &= 1(\frac{1}{6} - \frac{1}{5}e^{-t} + \frac{1}{30}e^{-6t}) - 1(\frac{1}{6} - \frac{1}{5}e^{-(t-5)} + \frac{1}{30}e^{-6(t-5)}) + 1(\frac{8}{5}e^{-t} - \frac{3}{5}e^{-6t}) \\ &= \frac{1}{5}(-e^{-t} + e^{-(t-5)}) + \frac{1}{30}(e^{-6t} - e^{-6(t-5)}) + \frac{8}{5}e^{-t} - \frac{3}{5}e^{-6t} \\ &= -\frac{17}{30}e^{-6t} - \frac{1}{30}e^{-6(t-5)} + \frac{7}{5}e^{-t} + \frac{1}{5}e^{-(t-5)} \end{aligned}$$

Problem 3

The dynamic system is given by:

$$\ddot{x}(t) + 2\dot{x}(t) + x(t) = -e^{-t} sint$$

where the initial conditions are: $x(0) = 0$ and $\dot{x}(0) = 2$.

$$\mathcal{L}\big[\dot{x}(t)\big] = sX(s) - x(0) = sX(s)$$
$$\mathcal{L}\big[\ddot{x}(t)\big] = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s) - 2$$
$$\mathcal{L}\big[-e^{-t}sint\big] = \frac{-1}{(s+1)^2+1} = -\frac{1}{s^2+2s+2}$$

Hence:

$$s^2X(s) - 2 + 2sX(s) + X(s) = -\frac{1}{s^2+2s+2}$$
$$X(s) = \frac{2s^2+4s+3}{(s+1)^2(s^2+2s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+2s+2}$$

The residues A, B, C, and D can be calculated as follows:

$$A = \lim_{s\to-1} \frac{d}{ds}\{(s+1)^2X(s)\} = 0$$
$$B = \lim_{s\to-1} \{(s+1)^2X(s)\} = 1$$

Doing a partial fraction expansion:

$$2s^2+4s+3 = A(s+1)(s^2+2s+2) + B(s^2+2s+2) + (Cs+D)(s+1)^2$$
$$2s^2+4s+3 = (s^2+2s+2) + (Cs+D)(s+1)^2$$
$$s^2+2s+1 = Cs^3 + 2Cs^2 + Cs + Ds^2 + 2Ds + D$$

We obtain:

$$s^3 : 0 = C$$
$$s^2 : 1 = 2C + D \to D = 1$$

Finally, the inverse laplace transform of $X(s)$ gives the time-domain function $x(t)$:

$$X(s) = \frac{1}{(s+1)^2} + \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2} + \frac{1}{(s+1)^2+1}$$
$$x(t) = \mathcal{L}^{-1}\big[X(s)\big] = te^{-t} + e^{-t}sint \qquad for \quad t \geq 0$$

Problem 4

a)

$$U(s) = \frac{1}{(s+2)}$$
$$Y(s) = \frac{2}{(s+3)^2(s+2)} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2} + \frac{C}{(s+2)}$$

Which will give the following terms in the time-domain: Ae^{-3t}, Bte^{-3t} and Ce^{-2t}, with $A = B = -2$, $C = 2$.

$$y(t) = 2e^{-2t} - 2e^{-3t} - 2te^{-3t} \qquad for \quad t \geq 0$$

b)

$$G(s) = \frac{2}{(s+3)^2} \;\rightarrow\; g(t) = \epsilon(t)2te^{-3t}$$
$$y(t) = \int_0^t u(\tau)g(t-\tau)d\tau = \int_0^t u(t-\tau)g(\tau)d\tau = \int_0^t e^{-2(t-\tau)}2\tau e^{-3\tau}d\tau =$$
$$= 2e^{-2t}\int_0^t \tau e^{-\tau}d\tau \overset{\text{int. by parts}}{=} 2e^{-2t}(\tau\int_0^t e^{-\tau} + \int_0^t e^{-\tau}d\tau)$$
$$= 2e^{-2t}(-\tau e^{-\tau}|_0^t + -e^{-\tau}|_0^t) = 2e^{-2t}(-te^{-t} - e^{-t} + 1)$$
$$= 2[e^{-2t} - e^{-3t} - te^{-3t}] \qquad for \quad t \geq 0$$

c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{(s+3)^2} = \frac{2}{s^2+6s+9} \;\rightarrow\; s^2Y(s) + 6sY(s) + 9Y(s) = 2U(s)$$

Assuming null initial conditions, the inverse Laplace transform will give the following differential equation:

$$\ddot{y} + 6\dot{y} + 9y = 2u \qquad y(0) = 0 \qquad \dot{y}(0) = 0$$

The state-space representation is obtained by defining the state variables as $x_1 = y$ and $x_2 = \dot{y}$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -6x_2 - 9x_1 + 2u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$